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Delocalization phenomena in one-dimensional models with long-range correlated disorder: a perturbative approach

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Abstract

We study the nature of electronic states in one-dimensional continuous models with weak correlated disorder. Using a perturbative approach, we compute the inverse localization length (Lyapunov exponent) up to terms proportional to the fourth power of the potential; this makes it possible to analyse the delocalization transition which takes place when the disorder exhibits specific long-range correlations. We find that the transition consists of a change of the Lyapunov exponent, which switches from a quadratic to a quartic depending on the strength of the disorder. Within the framework of the fourth-order approximation we also discuss the different localization properties which distinguish Gaussian from non-Gaussian random potentials.

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1. Introduction

In recent years, the interest for one-dimensional (1D) models with correlated disorder has been steadily growing, as it has become progressively clear that correlations of the random potential can deeply affect the structure of the electronic eigenstates, and endow 1D disordered models with far richer transport properties than it was previously thought.

For a long time it was believed that 1D systems could not display complex features such as the metal–insulator transition which takes place in three-dimensional (3D) models, since it was known that all one-electron states are localized in 1D systems with totally random potentials, regardless of the strength of disorder (see [1] and references therein). Further research, however, showed that 1D models include a fairly large variety of physical situations. On the one hand, deterministic quasi-periodic potentials were considered, which can generate localized or extended electronic states depending on the value of their parameters [2]. On the other hand, within the field of random 1D systems (as opposed to pseudo-random), some variants of the standard 1D Anderson model were found which exhibit a *discrete* set of extended states [3]. These systems are characterized by random potentials with *short-range* spatial correlations, a feature absent from the original Anderson model where the site energies are totally uncorrelated. Experimental evidence of the delocalization effect produced by these short-range correlations was recently found in semiconductor superlattices [4].

Eventually, the role of *long-range* correlations was also investigated. In [5] de Moura and Lyra considered a version of the discrete Anderson model with self-affine potential: specifically, the site energies of their model were generated by considering the potential as the trace of a fractional Brownian motion and imposing a normalization condition that kept fixed the amplitude of the potential fluctuations for all system sizes. Analysing this model, the authors showed how long-range correlated sequences of site energies could give rise to a *continuum* of extended states (although these results stirred some controversy [6]; see also the discussion in [7] where a somewhat different model was considered, with fluctuations of the potential increasing with the system size).

An important contribution to the understanding of the effect of long-range correlations of the disorder in the Anderson model was provided in [8], where the authors managed to establish an analytical relationship between localization length and potential pair correlators, and used this result to show how specific long-range disorder correlations lead to the appearance of '*mobility edges*' in 1D discrete models. An experimental confirmation of these findings was obtained by studying the transmission of microwaves in a single-mode waveguide with a random array of correlated scatterers [9]. The results established in [8] for discrete lattices were subsequently extended to continuous models and related to parallel phenomena occurring in different fields such as the propagation of waves in random media [10] and the dynamics of classical stochastic oscillators [11].

The discovery that 1D disordered systems can display a metal–insulator transition analogous to the one which takes place in 3D models constituted a crucial advancement for the understanding of anomalous transport properties of 1D random models. This theoretical progress is also relevant from the technological point of view, because it paves the way for the construction of 1D devices with pre-determined mobility edges which can be used as window filters in electronic, acoustic and photonic structures [12]. The importance of the result makes it therefore highly desirable to reach a complete comprehension of the link between long-range correlations of the disorder and the appearance of a continuum of extended electronic states in 1D models.

At the analytical level, our understanding of the 'delocalization transition' in 1D models rests on the results which were first derived in [8]. Unfortunately, both this work and those which have followed in its wake suffered from two main limitations: the disorder was supposed to be weak, so that a perturbative approach could be applied, and all the analytical results were obtained in the second-order approximation. Analytical calculations are greatly simplified by truncating the expansion of the inverse localization length to the second-order term; the main drawback of this choice is that it leaves open the question of the true nature of the states of the 'extended' phase. In fact, a vanishing second-order inverse localization length can be related to various physical phenomena: it can indicate that the electronic states are completely delocalized, but it can also be a sign that the states of the extended phase undergo a power-law localization (evidence for a behaviour of this kind, for instance, has been found at the mobility edge in specific quasi-periodic systems [13]). Finally, the inverse localization length may be non-zero even if its second-order part vanishes, in which case the 'extended' states would still be exponentially localized, although on a much larger spatial scale than that characterizing the states of the 'localized' phase.

The present work constitutes an attempt to partially remove the limits of the previous analyses; in this study we still focus our attention on the case of *weak* disorder but, with the help of a systematic perturbative technique, we manage to go beyond the second-order approximation and obtain analytical results correct to the fourth order of perturbation theory. In this way we can shed light on the true nature of the states that are classified as 'extended' in the framework of the second-order theory and we are also able to discuss the differences which emerge at this refined level of description between Gaussian and non-Gaussian random potentials.

In the case of Gaussian disorder with long-range correlations, it turns out that the electronic states are exponentially localized on both sides of the mobility edge identified in [8]; however, the dependence of their inverse localization length on the disorder strength changes from a *quadratic* to a *quartic* form upon crossing the critical threshold. For weak disorder, this implies that electrons in the 'extended' states can actually move over much longer distances than the electrons which find themselves in a 'localized' state. The use of terms such as 'mobility edge' and 'delocalization transition' is therefore legitimate, provided that one keeps in mind that the qualification of 'extended' must be understood as 'localized on a large spatial scale' rather than 'completely delocalized'. The analysis of the localization length in the fourth-order approximation also reveals that, upon increasing the energy of the electrons, a second threshold appears, which separates the states whose spatial extension scales with the inverse of the fourth power of the potential from states that are characterized by an even weaker localization (whose exact form cannot be ascertained within the limits of the fourth-order approximation).

The distinction between Gaussian and non-Gaussian disorder cannot be discussed within the framework of the second-order approximation, because the second-order inverse localization length only depends on the second moment of the random potential, and differences between Gaussian and non-Gaussian distributions only show up in the higher moments. A systematic computation of the higher order terms of the inverse localization length, however, establishes a connection between the *n*th term of the expansion and the corresponding *n*th moment of the potential; therefore, the fourth-order results obtained in this work allow a first analysis of the differences between Gaussian and non-Gaussian disorder. Lifting the Gaussian requirement seems to strengthen the localization of electronic states, because the fourth-order inverse localization length is increased by a new term proportional to the fourth *cumulant* of the random potential. Specifically, we analyse in detail the difference between a Gaussian potential with long-range binary correlations and a non-Gaussian potential with the same pair correlators but with a fourth-order cumulant having exponential decreasing form: we find that the non-zero cumulant produces quartic localization of *all* electronic states that lie beyond the second-order mobility edge, thus wiping out the additional threshold that appears in the Gaussian case.

This paper is organized as follows: in section 2 we give an exact formulation of the problem and define the model under study; in section 3 we expose the perturbative method used and give the general results for the second- and fourth-order terms of the localization length. We devote section 4 to the case of random potential with short-range correlations; the problem of long-range correlation is then discussed in section 5. Finally, we summarize the main results and express our conclusions in section 6.

2. Formulation of the problem

2.1. Definition of the model

We consider the 1D disordered model defined by the Schrödinger equation

$$-\frac{\hbar^2}{2m}\psi''(x) + \varepsilon U(x)\psi(x) = E\psi(x) \qquad (1)$$

where $\psi(x)$ represents the state of a quantum particle ('electron') of mass m and energy E moving in a continuous random potential $U(x)$. The dimensionless parameter ε is introduced to measure the strength of the disorder; for the sake of simplicity, in the rest of this paper we will adopt a system of units in which $\hbar^2/2m = 1$. We will consider electrons of positive energy, so that we can write $E = k^2$; since the wave vector k enters model (1) only via the energy, we can restrict our attention to the case $k > 0$ without loss of generality.

We define the statistical properties of the disorder through the moments of the potential $U(x)$. We assume that the model under study is spatially homogeneous in the mean; as a consequence, the nth moment of the potential

$$\chi_n(x_1, x_2, \ldots, x_{n-1}) = \langle U(x)U(x+x_1)\cdots U(x+x_{n-1})\rangle$$

depends only on $n-1$ relative coordinates. Here and in the following, we use angular brackets $\langle\cdots\rangle$ to denote the average over disorder realizations. We also want the average features of model (1) to be invariant under a change of sign of the potential; therefore we assume that all moments of odd order vanish. Finally, for reasons of mathematical convenience, we will suppose that a finite length scale l_c exists for model (1) such that the statistical correlations between the values $U(x_1)$ and $U(x_2)$ of the potential become negligible when the distance separating the points x_1 and x_2 exceeds l_c. For the particular case of the two-point correlation function, this assumption translates in the condition

$$\langle U(x_1)U(x_2)\rangle \simeq 0 \qquad \text{for} \quad |x_1 - x_2| \gg l_c. \tag{2}$$

More generally, the existence of a correlation length l_c implies that the n-point correlation functions $\langle U(x_1)U(x_2)\ldots U(x_n)\rangle$ must satisfy the so-called 'product property', meaning that, if the points $x_1, x_2, \ldots, x_n$ can be divided into two groups with $|x_i - x_j| \gg l_c$ for x_i and x_j belonging to different groups, the average $\langle U(x_1)U(x_2)\cdots U(x_n)\rangle$ takes the value obtained by averaging the two different groups separately. To deal with the case of long-range correlations, we will first make use of the previous hypothesis and derive results valid for any finite l_c; when possible, we will then take the limit $l_c \to \infty$.

Having defined the statistical properties of the random potential, we can proceed to enunciate the most specific assumption about the model under investigation. In the following, we will restrict our considerations to the case of *weak* disorder, as defined by the relation

$$\varepsilon^2\langle U(x)U(x')\rangle \ll E^2 \tag{3}$$

and by the analogous conditions on the higher moments of the random potential. Note that, to ensure that condition (3) is valid independently of the specific form of the function $U(x)$, the parameter ε must satisfy the condition $\varepsilon \ll 1$.

Equation (1), together with the specified statistical properties of the random potential and the assumption of weak disorder, completely defines the model under study. Our goal consists in determining the spatial behaviour of the solutions of equation (1); to this end it is necessary to give a precise definition of the key parameter known as localization length.

2.2. Localization length

To shed light on the localization properties of model (1), we will use as indicator of the localized or extended nature of the electronic states the quantity:

$$\lambda = \lim_{x\to\infty} \frac{1}{4x} \ln\langle \psi^2(x)k^2 + \psi'^2(x)\rangle. \tag{4}$$

In definition (4) the wavefunction $\psi(x)$ can be supposed to be real, so that no need arises to consider the complex extension of the logarithm function.

In this paper, we will always refer to the quantity (4) as the *inverse localization length* or *Lyapunov exponent*, although this parameter is more usually defined through the expression

$$\tilde{\lambda} = \lim_{x\to\infty} \frac{1}{2x} \langle \ln(\psi^2(x)k^2 + \psi'^2(x)) \rangle. \tag{5}$$

The parameters (4) and (5) differ in principle, but both can be used effectively to ascertain whether the electronic states are exponentially localized or not; in fact, they belong to the same family of generalized Lyapunov exponents

$$L_q = \lim_{x\to\infty} \frac{1}{x} \ln(\langle |\psi(x)|^q \rangle^{1/q})$$

which were introduced long ago for the study of localization in 1D disordered models (see [14] and references therein). In technical terms, equation (5) defines the standard Lyapunov exponent L_0, whereas the parameter (4) is equal to the 'generalized Lyapunov exponent of order two' divided by a factor 2, $\lambda = L_2/2$. The exact relation between the Lyapunov exponents (4) and (5) is not a completely solved problem; however, it is clear that both indicators can be used to determine whether an eigenfunction of equation (1) is extended, because in this case both parameters vanish. As for exponentially localized states, for the case of interest here, i.e. that of weak disorder, one can prove that the Lyapunov exponents (4) and (5) coincide at least to the second order of perturbation theory [11]. In section 4 we will come back to this point to show that at least in a special case the identity holds also beyond the second-order approximation.

From the physical point of view, the difference between the alternative parameters (4) and (5) is perhaps best understood if the problem of electronic localization in model (1) is redefined in terms of the dynamics of a stochastic oscillator. In fact, the physical comprehension of model (1) can be enhanced by observing that the stationary Schrödinger equation (1) is completely equivalent to the dynamical equation

$$\ddot{q}(t) + k^2(1 - \varepsilon U(t)/k^2)q(t) = 0 \tag{6}$$

which determines the evolution of an oscillator with frequency k perturbed by the noise $-U(t)/k^2$. Underlying the mathematical equivalence of equations (1) and (6) is the physical parallelism between the localization of the quantum states of model (1) and the energetic instability of the stochastic oscillator (6) [11]. The connection between the two phenomena emerges clearly if one writes the inverse localization length (4) in terms of the dynamical variables of the random oscillator. The Lyapunov exponent (4) can then be expressed as

$$\lambda = \lim_{t\to\infty} \frac{1}{4t} \ln\langle k^2(p^2(t) + q^2(t))\rangle$$

and interpreted as the growth rate of the average energy of the random oscillator (6). A positive value of the parameter (4) can therefore be read both as a sign of exponential localization of the eigenstates of model (1) and as an indication that the stochastic oscillator (6) is energetically unstable. As for the parameter (5), it can be rewritten in terms of the oscillator variables as

$$\tilde{\lambda} = \lim_{T\to\infty} \lim_{\delta\to 0} \int_0^T \ln \frac{q(t+\delta)}{q(t)} \, \mathrm{d}t.$$

In this form, the parameter (5) reveals itself as the exponential divergence rate of nearby orbits, i.e. as the Lyapunov exponent for the trajectories of the stochastic oscillator. We can conclude that the parameters (4) and (5) are both growth rates, measuring respectively the increase of the oscillator energy and that of the distance between initially nearby orbits. The connection between orbit instability and energetic instability explains why both (4) and (5) can be used as indicators of the localized nature of the eigenstates of model (1). Our choice of the definition (4) for the inverse localization length is justified by the fact that it allows for the mathematically simple and physically transparent analytical treatment described in the next section.

A final remark is in order here. Lyapunov exponents of the form (4) as well as (5) are completely adequate choices for the study of localization phenomena only if exponential localization of the electrons is expected. When the electronic states exhibit a more complicated behaviour (like power-law localization) other forms of localization length may be opportunely used to characterize the shape of the electronic wavefunctions [13, 15]. The primary aim of this work, however, was to ascertain whether the states beyond the second-order mobility edge were exponentially localized or not and the Lyapunov exponent (4) is a sensitive enough indicator for this purpose.

3. Perturbative expansion and general results

As a first step to compute the Lyapunov exponent (4), we derive a differential equation for $\psi^2(x)$ and $\psi'^2(x)$. Using equation (1) as a starting point, it is easy to show that the vector

$$u(x) = \begin{pmatrix} \psi^2(x) \\ \psi'^2(x)/k^2 \\ \psi(x)\psi'(x)/k \end{pmatrix}$$

obeys the equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} = (\mathbf{A} + \varepsilon\xi(x)\mathbf{B})u \tag{7}$$

where $\xi(x)$ is the scaled potential $\xi(x) = -U(x)/k^2$ and the symbols $\mathbf{A}$ and $\mathbf{B}$ stand for the matrices

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 2k \\ 0 & 0 & -2k \\ -k & k & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2k \\ -k & 0 & 0 \end{pmatrix}.$$

At this point, our problem consists of extracting information on the average vector $\langle u(x)\rangle$ from the stochastic equation for $u(x)$; this can be done using a method developed by Van Kampen [16], which replaces equation (7) with an ordinary differential equation of the form

$$\frac{\mathrm{d}\langle u(x)\rangle}{\mathrm{d}x} = \mathbf{K}(x)\langle u(x)\rangle. \tag{8}$$

The generator $\mathbf{K}(x)$ in equation (8) is a sure operator that can be expressed in terms of a cumulant expansion

$$\mathbf{K}(x) = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{K}_n(x) \tag{9}$$

where the partial generators $\mathbf{K}_n(x)$ are functions of specific combinations (known as 'ordered cumulants') of the moments of the random matrix

$$\mathbf{M}(x) = \xi(x)\exp(-\mathbf{A}x)\mathbf{B}\exp(\mathbf{A}x). \tag{10}$$

Van Kampen [16] and Terwiel [17] have established a set of systematic rules for constructing all the terms of the series (9); applying their prescriptions, one obtains that all generators of odd order are zero because the odd moments of the potential vanish. As for the even-order generators, the zeroth-order term has the simple form $\mathbf{K}_0 = \mathbf{A}$, which corresponds to ignoring the random potential. The first term where disorder manifests its effect is the second-order generator

$$\mathbf{K}_2(x) = \int_0^x \mathrm{d}x_1 \exp(\mathbf{A}x)\langle \mathbf{M}(x)\mathbf{M}(x_1)\rangle \exp(-\mathbf{A}x)$$

which is proportional to the two-point correlator; a refinement of this result is obtained by taking into account the fourth-order term

$$\begin{aligned}\mathbf{K}_4(x) = \int_0^x \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \int_0^{x_2} \mathrm{d}x_3 \exp(-\mathbf{A}x)[&\langle \mathbf{M}(x)\mathbf{M}(x_1)\mathbf{M}(x_2)\mathbf{M}(x_3)\rangle \\ &- \langle \mathbf{M}(x)\mathbf{M}(x_1)\rangle\langle \mathbf{M}(x_2)\mathbf{M}(x_3)\rangle - \langle \mathbf{M}(x)\mathbf{M}(x_2)\rangle\langle \mathbf{M}(x_1)\mathbf{M}(x_3)\rangle \\ &- \langle \mathbf{M}(x)\mathbf{M}(x_3)\rangle\langle \mathbf{M}(x_1)\mathbf{M}(x_2)\rangle] \exp(\mathbf{A}x). \end{aligned} \tag{11}$$

The integrand in equation (11) is a non-trivial example of *ordered cumulant* of the matrix (10); note that it differs from an ordinary cumulant because the matrices $\mathbf{M}(x_1)$ and $\mathbf{M}(x_2)$ do not commute for $x_1 \neq x_2$. Although the rules of Van Kampen make it possible to derive every partial generator $\mathbf{K}_n(x)$, for the purpose of this work we can truncate the series (9) and neglect all terms of order higher than four. As can be seen from equation (8), the behaviour of $\langle u(x)\rangle$ for $x \to \infty$ is determined by the eigenvalue with the largest real part of the asymptotic generator $\mathbf{K}(\infty) = \lim_{x\to\infty} \mathbf{K}(x)$; we are thus led to the conclusion that, in order to determine the Lyapunov exponent (4) with fourth-order accuracy, we have to solve the secular equation for the truncated asymptotic generator

$$\overline{\mathbf{K}}(\infty) = \mathbf{A} + \varepsilon^2 \mathbf{K}_2(\infty) + \varepsilon^4 \mathbf{K}_4(\infty). \tag{12}$$

Note that the existence of the asymptotic generators $\mathbf{K}_n(\infty)$ is mathematically ensured by the sufficient condition that the random potential $U(x)$ possesses a finite correlation length l_c. This implies that, when we relax this condition in order to study the case of long-range correlations, we will have to pay attention to whether the asymptotic generators continue to be well defined or not.

Solving the secular equation for the matrix (12), one obtains that the inverse localization length (4) can be written as

$$\lambda(k) = \varepsilon^2 \lambda_2(k) + \varepsilon^4 \left(\lambda_4^{(G)}(k) + \lambda_4^{(NG)}(k)\right) + o(\varepsilon^4) \tag{13}$$

where the second-order term has the standard form

$$\lambda_2(k) = \frac{1}{4k^2} \int_0^\infty \chi_2(x) \cos(2kx)\, \mathrm{d}x. \tag{14}$$

As for the fourth-order term of the Lyapunov exponent (13), its first component can be written as

$$\begin{aligned}\lambda_4^{(G)}(k) = \frac{1}{16k^5}&\varphi_s(k,0)[\varphi_c(k,0) + 4\varphi_c(0,0)] \\ &- \frac{1}{8k^4}[\varphi_c(k,0) + 2\varphi_c(0,0)] \int_0^\infty \chi_2(x) x \cos(2kx)\, \mathrm{d}x \\ &+ \frac{1}{8k^4}\varphi_s(k,0) \int_0^\infty \chi_2(x) x \sin(2kx)\, \mathrm{d}x \\ &+ \frac{1}{4k^4} \int_0^\infty \left[\varphi_c(0,x)\varphi_c(k,x) - \varphi_c^2(0,x)\cos(2kx)\right] \mathrm{d}x \end{aligned} \tag{15}$$

where the functions $\varphi_c(k,x)$ and $\varphi_s(k,x)$ are defined as

$$\begin{aligned}\varphi_c(k,x) &= \int_x^\infty \chi_2(y) \cos(2ky)\, \mathrm{d}y \\ \varphi_s(k,x) &= \int_x^\infty \chi_2(y) \sin(2ky)\, \mathrm{d}y. \end{aligned} \tag{16}$$

The second component turns out to be

$$\lambda_4^{(NG)}(k) = \frac{1}{k^4}\int_0^{\infty} \mathrm{d}x \int_0^{x} \mathrm{d}y \int_x^{\infty} \mathrm{d}z \Delta_4(y, x, z)[\cos(2ky)\cos(2kz - 2kx) - \cos(2kz)] \tag{17}$$

where the symbol $\Delta_4(x_1, x_2, x_3)$ represents the fourth cumulant of the random potential

$$\Delta_4(x_1, x_2, x_3) = \chi_4(x_1, x_2, x_3) - \chi_2(x_1)\chi_2(x_3 - x_2) - \chi_2(x_2)\chi_2(x_3 - x_1) - \chi_2(x_3)\chi_2(x_2 - x_1).$$

In contrast to the relative simplicity of the second-order result (14), the fourth-order terms (15) and (17) exhibit a non-trivial dependence on the binary correlator and on the fourth cumulant, respectively. The greater complexity of the results is a consequence of the more sophisticated description of localization, which is visualized as an interference effect generated by double scatterings of the electron in the second-order scheme, whereas the fourth-order approximation also takes into account quadruple scattering processes. The more accurate description of the localization processes makes it possible to draw a distinction between Gaussian and non-Gaussian disorders. These two classes of random potentials are necessarily undifferentiated within the framework of the second-order approximation, since the second-order Lyapunov exponent (14) depends only on the second moment of the potential; in the fourth-order description, however, the differences between the two kinds of potentials emerge in the component (17) which, being a function of the cumulant, vanishes if the disorder is Gaussian. In conclusion, the separation of the fourth-order term of the Lyapunov exponent in the two components (15) and (17) is justified by the distinct physical character of these two parts: while the former is common to all Gaussian and non-Gaussian potentials with the same two-point correlation function, the latter describes the specific effect of the non-Gaussian nature of the disorder.

4. Disorder with short-range correlations

We now apply the general formulae derived in the preceding section for the Lyapunov exponent to the case of a random potential with exponentially decaying correlations. Specifically, we assume that the two-point correlation function and the fourth-order cumulant have the form

$$\chi_2(x) = \sigma_2 \, \mathrm{e}^{-\beta_2|x|} \tag{18}$$

and

$$\Delta_4(x, y, z) = \sigma_4 \exp[-\beta_4(|x| + |y| + |z|)]. \tag{19}$$

In equations (18) and (19), the constants β_2^{-1} and β_4^{-1} represent the range of the correlation function and of the fourth cumulant, respectively; note that the two parameters may be different.

Inserting the binary correlator (18) in equation (14), one obtains that the second-order Lyapunov exponent is

$$\lambda_2(k) = \frac{\sigma_2}{4k^2}\frac{\beta_2}{4k^2 + \beta_2^2}. \tag{20}$$

This result confirms the general rule that random potentials with short-range correlations produce localization of all the electronic eigenstates (with the possible exception of a discrete set of extended states). Note that the Lyapunov exponent (20) tends to zero in the limit $\beta_2 \to 0$; physically, this means that the localization of the electronic states becomes weaker and weaker as the range of the disorder correlations stretches over increasingly larger distances.

The binary correlator (18) also determines the form of the Gaussian part of the fourth-order Lyapunov exponent. For $x > 0$ the functions (16) take the form

$$\varphi_c(k, x) = \frac{\sigma_2}{4k^2 + \beta_2^2}[\beta_2 \cos(2kx) - 2k \sin(2kx)] \exp(-\beta_2 x)$$
$$\varphi_s(k, x) = \frac{\sigma_2}{4k^2 + \beta_2^2}[2k \cos(2kx) + \beta_2 \sin(2kx)] \exp(-\beta_2 x).$$

Correspondingly, equation (15) becomes

$$\lambda_4^{(G)}(k) = \frac{\sigma_2^2}{4} \frac{\beta_2\left(\beta_2^4 + 22k^2\beta_2^2 + 48k^4\right)}{k^4\left(\beta_2^2 + k^2\right)\left(\beta_2^2 + 4k^2\right)^3}. \tag{21}$$

This result coincides with the expression obtained in [18] for the fourth-order term of the inverse localization length (5); an important consequence of equation (21), therefore, is that the identity of the localization lengths (4) and (5) is not restricted to the second order, but holds at least up to the fourth order when disorder is weak and Gaussian and the correlation function has the exponential form (18).

To complete the discussion of the fourth-order approximation, we have to compute the non-Gaussian term (17) when the fourth cumulant has the form (19). A straightforward calculation leads to

$$\lambda_4^{(NG)}(k) = \frac{15\sigma_4}{8} \frac{\beta_4}{k^2\left(\beta_4^2 + k^2\right)\left(\beta_4^2 + 4k^2\right)\left(9\beta_4^2 + 4k^2\right)}. \tag{22}$$

An important conclusion which can be drawn from equation (22) is that the non-Gaussian character of the disorder produces an enhancement of localization. In the present case, the increase is generally negligible (except in the limit case of $\beta_2 = 0$, which corresponds to a strongly correlated potential with a constant correlation function) because the second-order term (20) does not vanish as long as $\beta_2 > 0$. The non-Gaussian term (22) can become crucial, however, when long-range correlations of the potential come into play, as we will discuss in the next section.

5. Disorder with long-range correlations

In this section, we focus our attention on the case of disorder with two-point correlation function of the form

$$\chi_2(x) = \sigma_2 \frac{\sin(2k_c x)}{x} \tag{23}$$

which is known to make the second-order Lyapunov exponent vanish over a continuum of energy values [8]. To enhance the physical understanding of the problem, we will first analyse the physical properties which descend from the specific form (23) of the two-point correlation function, postponing the discussion of non-Gaussian effects until the end of this section.

The correlator (23) is substantially different from the function (18) considered in the previous section because there is no finite length scale beyond which it becomes negligible. As a consequence of the slow decay towards infinity of the function (23), great attention must be paid when applying the formalism developed in section 3 to the present case. In fact, in the derivation of the general results of section 3 we avoided any mathematical inconsistency by conveniently assuming the existence of a finite correlation length; one could therefore question the Chevalier application of the general formulae (14) and (15) to a case in which no correlation length can be properly individuated and condition (2) is not satisfied. To avoid

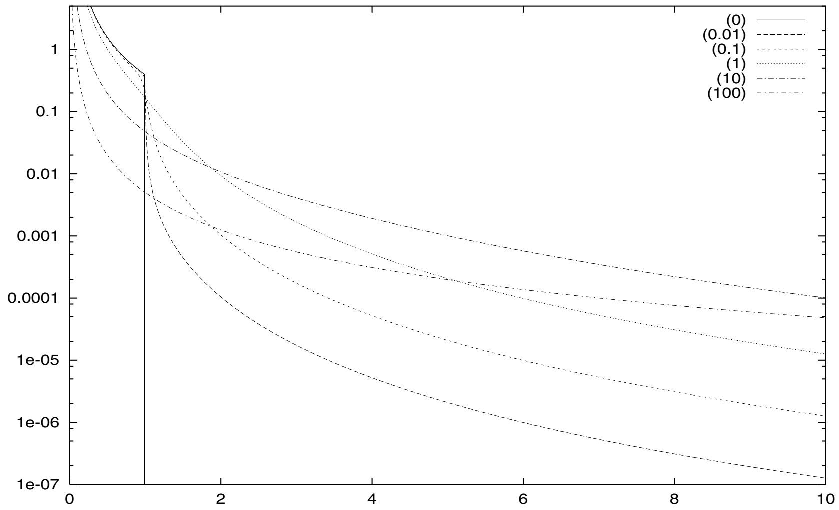

Figure 1. Second-order Lyapunov exponent $\log_{10}[\lambda_2(\beta, k)/\sigma_2]$ versus k/k_c. The values of β/k_c corresponding to each curve are reported in parentheses in the right upper corner of the figure.

these difficulties, we will study long-range correlators of the form (23) as the limit case for $\beta \to 0^+$ of the more general correlation function

$$\chi_2(x) = \sigma_2 \frac{\sin(2k_c x)}{x} \, \mathrm{e}^{-\beta x}. \tag{24}$$

Since the correlator (24) decays exponentially for $\beta > 0$, the results of section 3 can be safely used for this model; the localization properties for the particular case (23) can then be deduced by discussing the limit form of the Lyapunov exponent for $\beta \to 0^+$. This approach is generally successful, but it fails for $k = k_c$, because for this value of the wave vector the asymptotic limit of the generator (9) exists only as long as β is positive. Our results for the Lyapunov exponent, therefore, are not valid in a small neighbourhood of k_c.

Having defined our approach, we can proceed to derive the second-order Lyapunov exponent for the correlation functions (24) and (23). Inserting (24) in equation (14), one obtains

$$\lambda_2(\beta, k) = \frac{\sigma_2}{8k^2} \left\{ \arctan\left[\frac{4\beta k_c}{\beta^2 + 4\left(k^2 - k_c^2\right)} \right] + \pi\theta\left(4k_c^2 - 4k^2 - \beta^2\right) \right\} \tag{25}$$

where $\theta(x)$ is the step function defined as

$$\theta(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leqslant 0. \end{cases}$$

In the limit $\beta \to 0^+$ expression (25) becomes

$$\lambda_2(k) = \begin{cases} \dfrac{\sigma_2 \pi}{8k^2} & \text{for} \quad 0 < k < k_c \\ 0 & \text{for} \quad k_c < k. \end{cases} \tag{26}$$

The behaviour of the second-order Lyapunov exponents (25) and (26) is represented in figure 1.

Equation (26) shows that long-range correlations of the form (23) make the second-order Lyapunov exponent vanish when the wave vector k exceeds the critical value k_c. The physical reason of this behaviour is that a random potential with correlation function (23) has a spectral density which vanishes for $k > k_c$ (as follows from the Wiener–Khintchin theorem); as a consequence, electron waves with wave vectors larger than k_c cannot be backscattered in the first Born approximation. This implies that the interference effects which produce second-order localization are absent for $k > k_c$; the result (26) is thus explained. This connection between localization length and spectral properties of the potential explains why, within the vast class of long-range correlated potentials, we focus our attention on random potentials with correlation function of the specific form (23).

The result (26) was interpreted in [8] as an evidence that specific long-range correlated potentials can create a continuum of extended states and produce mobility edges even in 1D models. This conclusion, although of relevant theoretical and practical importance, is not completely satisfactory, because the fact that the second-order term (26) vanishes does not guarantee that the same be true for all higher order terms of the Lyapunov exponent. Consequently, it is impossible to ascertain on the basis of equation (26) whether the electronic states for $k > k_c$ are really extended or, rather, localized in a different form or on a much larger spatial scale than the states for $k < k_c$.

To shed light on this point, it is useful to determine the fourth-order Lyapunov exponent (15) in the case of long-range correlations of the form (23). Once more, we will first compute the fourth-order term (15) for the correlator (24) and then we will use this result to study the limit $\beta \to 0^+$. When the two-point correlation function has the form (24), the functions (16) become

$$\varphi_c(k,x) = \frac{\sigma_2}{2}\mathrm{Im}\{E_1[\beta x - \mathrm{i}2(k+k_c)x] - E_1[\beta x - \mathrm{i}2(k-k_c)x]\}$$
$$\varphi_s(k,x) = \frac{\sigma_2}{2}\mathrm{Re}\{-E_1[\beta x - \mathrm{i}2(k+k_c)x] + E_1[\beta x - \mathrm{i}2(k-k_c)x]\}$$

where $E_1(z)$ is the exponential integral defined in the complex plane by

$$E_1(z) = \int_z^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t \qquad \text{with} \quad |\arg z| < \pi$$

(see, e.g., [19]). Correspondingly, the Gaussian part of the fourth-order Lyapunov exponent takes the form

$$\begin{aligned}
\lambda_4^{(G)}(\beta,k) = \frac{\sigma_2^2}{4k^4}\Bigg\{&\frac{1}{16}\left[\frac{\beta}{\beta^2+4(k-k_c)^2} - \frac{\beta}{\beta^2+4(k+k_c)^2}\right.\\
&\left.+\frac{1}{k}\left(\alpha_1(k)+\frac{\pi}{2}s_3(k)\right)\right]\ln\frac{\beta^2+4(k+k_c)^2}{\beta^2+4(k-k_c)^2}\\
&-\frac{\beta}{8}\left[\frac{1}{\beta^2+4k_c^2}+\frac{1}{\beta^2+4(k+k_c)^2}\right]\ln\frac{\beta^2+(k+2k_c)^2}{\beta^2+k^2}\\
&-\frac{\beta}{8}\left[\frac{1}{\beta^2+4k_c^2}+\frac{1}{\beta^2+4(k-k_c)^2}\right]\ln\frac{\beta^2+(k-2k_c)^2}{\beta^2+k^2}\\
&-\frac{1}{2}\frac{k+k_c}{\beta^2+4(k+k_c)^2}\left[\alpha_1(k)+\alpha_3(k)+\frac{\pi}{2}s_3(k)\right]\\
&+\frac{1}{2}\frac{k-k_c}{\beta^2+4(k-k_c)^2}\left[\alpha_1(k)+\alpha_4(k)+\frac{\pi}{2}s_3(k)+\pi s_2(k)\right]\\
&+\frac{k_c}{\beta^2+4k_c^2}\left[2\alpha_1(k)-\alpha_2(k)+\pi s_3(k)-\frac{\pi}{2}s_1(k)\right]+F_\beta(k)\Bigg\}
\end{aligned} \qquad (27)$$

where the functions $\alpha_i(k)$ are defined as

$$\alpha_1(k) = \frac{1}{2}\arctan\frac{4\beta k_c}{\beta^2 + 4\left(k^2 - k_c^2\right)}$$
$$\alpha_2(k) = \frac{1}{2}\arctan\frac{4\beta k_c}{\beta^2 + k^2 - 4k_c^2}$$
$$\alpha_3(k) = \arctan\frac{2\beta k_c}{\beta^2 + k^2 + 2kk_c}$$
$$\alpha_4(k) = \arctan\frac{2\beta k_c}{\beta^2 + k^2 - 2kk_c}$$

and the symbols $s_i(k)$ represent the step-functions

$$s_1(k) = \begin{cases} 1 & \text{if} \quad \beta^2 + k^2 - 4k_c^2 < 0 \\ 0 & \text{otherwise} \end{cases}$$
$$s_2(k) = \begin{cases} 1 & \text{if} \quad \beta^2 + k^2 - 2kk_c < 0 \\ 0 & \text{otherwise} \end{cases}$$
$$s_3(k) = \begin{cases} 1 & \text{if} \quad \beta^2 + 4\left(k^2 - k_c^2\right) < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the function $F_\beta(k)$ which appears on the rhs of equation (27) is defined by the integral representation

$$F_\beta(k) = \frac{1}{\sigma_2^2 k}\int_0^\infty \chi_2(x)[\varphi_c(0,0) - \varphi_c(0,x)]\sin(2kx)\,\mathrm{d}x. \tag{28}$$

For the computation of the actual values of the function (28) it is convenient to use its representation in series form; the interested reader is referred to the appendix for a discussion of this point.

The expression of the fourth-order Lyapunov exponent (27) is quite complicated, but it simplifies significantly in the case of greatest interest, i.e. in the limit $\beta \to 0^+$, when it takes the form

$$\lambda_4^{(G)}(k) = \begin{cases} \Lambda_1(k) & \text{for} \quad 0 < k < k_c \\ \Lambda_2(k) & \text{for} \quad k_c < k < 2k_c \\ 0 & \text{for} \quad 2k_c < k \end{cases} \tag{29}$$

where the functions $\Lambda_1(k)$ and $\Lambda_2(k)$ are defined as

$$\Lambda_1(k) = \frac{\sigma_2^2}{4k^4}\left[\frac{\pi}{32k}\ln\left(\frac{k+k_c}{k-k_c}\right)^2 - \frac{\pi}{8}\frac{k^2 + kk_c + k_c^2}{k_c\left(k_c^2 - k^2\right)} + F_0(k)\right]$$
$$\Lambda_2(k) = \frac{\sigma_2^2}{4k^4}\left[\frac{\pi}{8}\frac{2k_c - k}{k_c(k - k_c)} + F_0(k)\right].$$

The behaviour of the fourth-order Lyapunov exponent (27) and of its limit form (29) is represented in figures 2 and 3. As shown in the figures, for large values of β the fourth-order term (27) is a smooth and decreasing function of k, which assumes small but positive values everywhere. In this case, the fourth-order component (27) is a negligible correction to the second-order term (25) of the Lyapunov exponent. The behaviour of $\lambda_4^{(G)}(\beta, k)$ becomes more complex for small values of β. First, the function loses its monotonic behaviour and even starts to assume negative values in an interval comprised within the range $0 < k < k_c$. Note

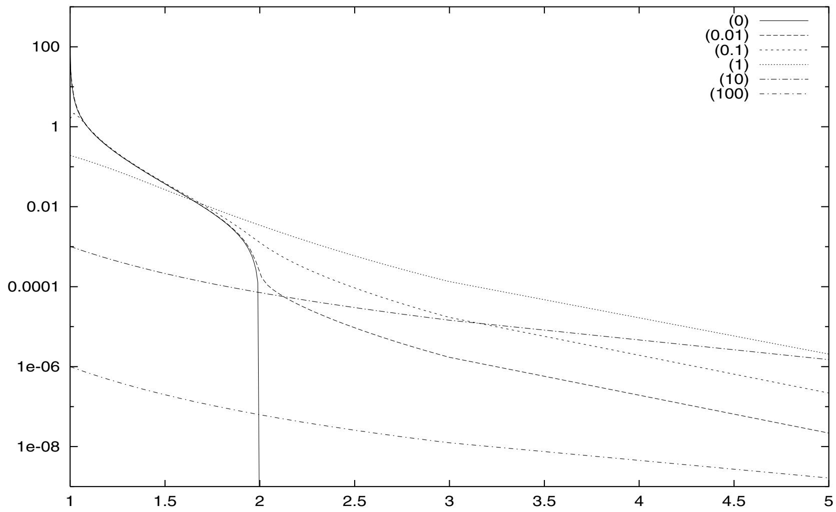

Figure 2. Fourth-order Lyapunov exponent $\log_{10}[\lambda_4^{(G)}(\beta, k)/\sigma_2^2]$ versus k/k_c. The values of β/k_c corresponding to each curve are reported in parentheses.

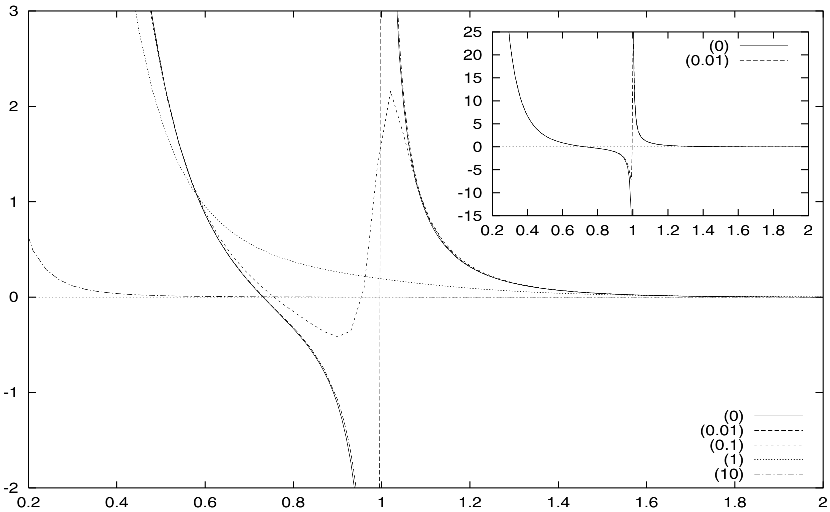

Figure 3. Fourth-order Lyapunov exponent $\lambda_4^{(G)}(\beta, k)/\sigma_2^2$ versus k/k_c. The values of β/k_c corresponding to each curve are reported in parentheses. The inset shows the divergence for $\beta = 0$ and the pronounced peaks for $\beta = 10^{-2}\, k_c$.

that the fourth-order term (27) can become negative as long as $\lambda_2(k)$ is positive; physically, this means that the fourth-order correction reduces the localization effect. Finally, for $\beta \ll k_c$, the function $\lambda_4^{(G)}(\beta, k)$ develops a pronounced negative minimum in a left neighbourhood of k_c and a sharp positive peak in a right neighbourhood k_c. For values of k close to k_c, therefore, the fourth-order term can represent a conspicuous correction to the leading term when $\beta \ll k_c$; since in this limit the second-order term tends to assume the discontinuous form (26), one can conclude that the effect of the fourth-order correction is that of smoothing the Lyapunov exponent in the region $k \simeq k_c$.

In the limit case $\beta = 0^+$, the fourth-order Lyapunov exponent (29) displays a qualitative behaviour close to that of the corresponding term (27) for small values of β. There are two main differences though: first, the fourth-order Lyapunov exponent (29) is strictly zero for $k > 2k_c$ and, second, it diverges for $k \to k_c$. This divergence must be disregarded, because, as already observed, the asymptotic generator (12) is not defined for $k = k_c$ in the limit $\beta \to 0^+$. Having established the range of validity of expression (29), we can now comment on the physical meaning of the result. The first remarkable aspect of the fourth-order Lyapunov exponent (29) is that it takes positive values in the interval $k_c < k < 2k_c$ where the second-order term (26) vanishes. This means that the inverse localization length, which is a quadratic function of the potential for $k < k_c$, assumes a quartic form when the wave vector exceeds the critical value k_c. This feature clarifies the nature of the transition which occurs at $k = k_c$: when the threshold is crossed, the electronic states continue to be exponentially localized, but over spatial scales which are much larger than those typical of the states with $k < k_c$. The transition at k_c, therefore, does not bring a qualitative change from exponential localization to complete delocalization, but only a quantitative increase of order $O(1/\varepsilon^2)$ in the spatial extension of the electronic states. Since we are considering the case of weak disorder, $\varepsilon \ll 1$, this increase can be huge; in loose terms, one can therefore speak of 'delocalization transition' once it is clear that the 'extended' states are not completely delocalized like Bloch states in a crystal lattice. A second relevant feature of the fourth-order Lyapunov exponent (29) is that it vanishes for $k > 2k_c$ so that, in the case of Gaussian disorder, a second transition takes place at $k = 2k_c$, with the electrons becoming even more weakly localized. The spatial extension of the electronic states beyond this second threshold cannot be estimated within the fourth-order approximation, which allows the only conclusion that for $k > 2k_c$ the inverse localization length must be an infinitesimal of order $O(\varepsilon^6)$ or higher.

So far our discussion has been centred on the term (15), which allows for a complete analysis of Gaussian potentials. For non-Gaussian disorder, however, one must also take into account the cumulant-generated component (17) of the Lyapunov exponent, which has the effect of increasing localization. The specific form of the non-Gaussian term (17) depends on that of the fourth cumulant of the disorder; to discuss a concrete example, we will consider the case of a random potential with correlation function and fourth cumulant of the forms (23) and (19), respectively. In this case, the fourth-order term of the Lyapunov exponent is the sum of (29) and of the additional non-Gaussian component (22). Since the extra term (22) is always positive, all electronic states with $k > k_c$ are exponentially localized with inverse localization length proportional to the fourth power of the disorder strength, $\lambda(k) \propto \varepsilon^4$. We can therefore conclude that, in the case of disorder with long-range correlations of the form (23), non-Gaussian potentials generate a stronger localization of the electrons than their Gaussian counterparts.

6. Conclusions

In this paper, we studied the localization properties of 1D models with weak disorder, focusing our attention on the role played by spatial correlations of the random potential in shaping the structure of the electronic states. Using a perturbative approach, we were able to derive the standard second-order expression for the inverse localization length and the fourth-order correction. We used this result to investigate the delocalization transition which takes place when the disorder exhibits specific long-range correlations of the form (23). The analysis of section 5 reveals that this transition consists of a sharp change of the scaling law for the inverse localization length: when the electronic wave vector reaches the critical value $k = k_c$, the Lyapunov exponent switches from a quadratic to a quartic dependence on the disorder strength ε and, correspondingly, the spatial extension of the electronic states increases by a factor $O(1/\varepsilon^2)$. In the case of Gaussian potentials, a second critical value of the wave vector exists at $k = 2k_c$, beyond which the electronic states become even more delocalized, with Lyapunov exponent $\lambda = o(\varepsilon^4)$ for $\varepsilon \to 0$. This additional transition is absent in the case of a non-Gaussian potential characterized by the long-range correlation function (23) and by an exponentially decaying fourth cumulant of the form (19); in fact, in this case all electronic states for $k > k_c$ are localized with a Lyapunov exponent proportional to the fourth power of the potential strength. It seems therefore that non-Gaussian potentials may produce stronger localization effects than their Gaussian counterparts.

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Appendix. Series representations of the function $F_\beta(k)$

In this appendix, we give two representations in series form for the function $F_\beta(k)$ defined in integral form in equation (28). For $k > 2k_c$ the function (28) can be expressed through the series

$$F_\beta(k) = \frac{1}{2k}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left\{\left[\frac{\beta^2+4k_c^2}{\beta^2+4(k+k_c)^2}\right]^{\frac{n}{2}}\cos(n\delta_1) - \left[\frac{\beta^2+4k_c^2}{\beta^2+4(k-k_c)^2}\right]^{\frac{n}{2}}\cos(n\delta_2)\right\}\sin(n\delta_0) \tag{30}$$

where

$$\delta_0 = \arctan\frac{2k_c}{\beta} \qquad \delta_1 = \arctan\frac{2(k+k_c)}{\beta} \qquad \delta_2 = \arctan\frac{2(k-k_c)}{\beta}.$$

For $k < \sqrt{k_c^2+\beta^2} - k_c$, the function $F_\beta(k)$ can be represented in the alternative form

$$F_\beta(k) = \frac{1}{4k}\arctan\left(\frac{2k_c}{\beta}\right)\ln\frac{\beta^2+4(k+k_c)^2}{\beta^2+4(k-k_c)^2} + G_\beta(k)$$

where $G_\beta(k)$ is the series

$$G_\beta(k) = \frac{1}{2k}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}c_n\left\{\left[\frac{(k+k_c)^2}{\beta^2+k_c^2}\right]^n - \left[\frac{(k-k_c)^2}{\beta^2+k_c^2}\right]^n\right\} \tag{31}$$

with coefficients

$$c_n = \sum_{l=0}^{\infty}\frac{1}{2n+l}\left(\frac{\beta}{2\sqrt{\beta^2+k_c^2}}\right)^l \sin\left[(2n+l)\arctan\left(\frac{k_c}{\beta}\right)\right].$$

Note that, when $\beta \geqslant \sqrt{8}k_c$, the convergence regions of the series (30) and (31) overlap, so that there is no need to use the integral representation (28) to compute the values of the function $F_\beta(k)$; the integral (28) must be evaluated only if $\beta < \sqrt{8}k_c$ and for those values of k comprised in the interval $\sqrt{k_c^2+\beta^2} - k_c < k < 2k_c$.

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